Fitting a Ladder Round a Corner

What is the longest ladder which will fit round a corner? We'll investigate this problem with Geometry Expressions. Along the way, we'll learn what an envelope curve is, and we'll see a great example of the relationship between solving an algebraic equation and lying on a curve. You'll need:

- Geometry Expressions
- An Algebra system

We'll assume the two corridors are of different widths x and y. To model this in Geometry Expressions, create a pair of infinite lines which will be one pair of walls. Constrain one to be the line Y=0, and the other to be the line X=0. Now draw in the other two walls of the corridor, constrain one to be distance x from the y axis, and the other to be distance y from the x axis. Draw the ladder DE, making sure D lies on the y axis and E lies on the x axis. Constrain the length of the ladder to be L.

(Figure 1).

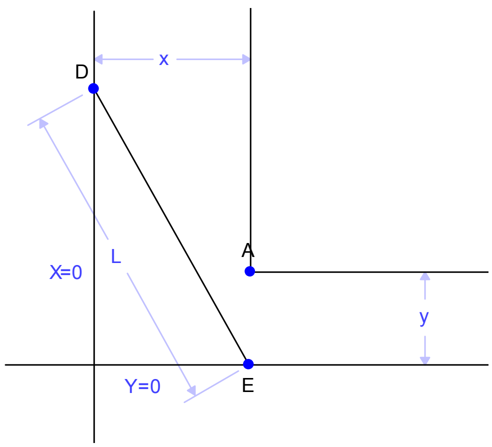

Figure 1:Moving a ladder round a corner

Try dragging E – do you notice that the length of the ladder changes? Now try locking L and then dragging E. Did your ladder fit round the corner? Try stretching or shrinking L using the slider bar, then dragging E again. Can you find a value for L which only just fits round the corner?

Equal Width Corridors

Before we solve the general problem, we will try a simpler version where the two corridors are the same width. You can make the corridors the same width in your drawing just by editing the length y and changing its value to x.

As before experiment till you find a value of L which only just fits. Now create the midpoint of L. What do you notice as you drag E?

In figure 2, we create the locus of the midpoint. To create a locus in Geometry Expressions, you need a parameter. A parameter is easily supplied by constraining point E to be distance t from the y-axis.

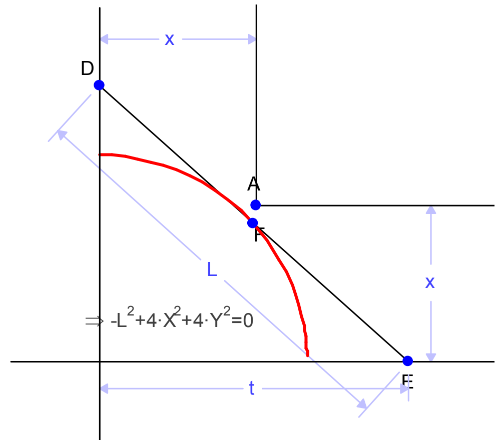

Figure 2: Locus of the midpoint of the ladder

Create the equation of the locus. What kind of curve is this?

For what length L would this curve just touch A?

Try replacing L by this value and verify that for any value of x, the ladder just fits round the corner.

Unequal Width Corridors

Now make your corridors unequal width again, and make the ladder length L.

Is the following statement true or false (look at Figure 3)?

"The ladder only just fits when the locus of center of the ladder passes through A"

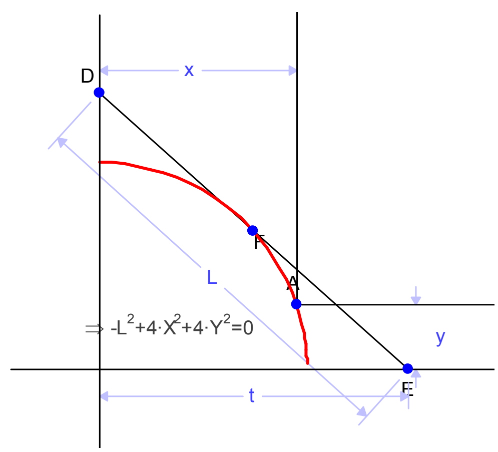

Figure 3: The locus of the midpoint along with corridors of unequal width.

False, right? In figure the ladder clearly does not fit round the corner, but the locus passes through A.

If the ladder only just fits, some part of the ladder must touch A as it moves, but we don't know which.

For this problem, instead of the locus of a specific point on the ladder we need the "envelope" of the whole ladder. You can create this curve by selecting the segment DE and then the locus tool (Figure 4).

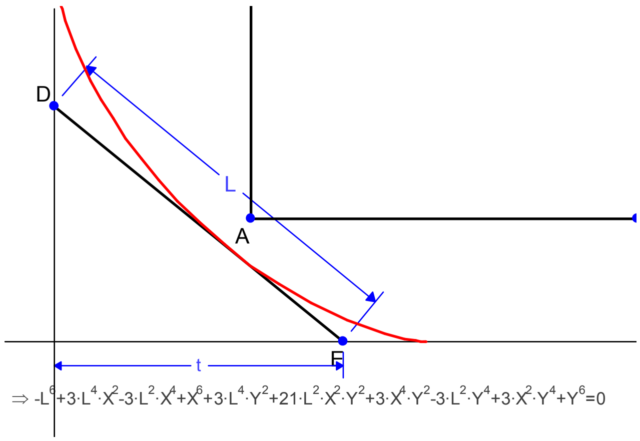

Figure 4: Envelope curve of the ladder, along with its equation

Again try dragging E, and observe how the ladder relates to the envelope curve.

Is the following statement true or false (look at Figure 3)?

"The ladder only just fits when the envelope curve of the ladder passes through A"

You can experiment by locking L so it doesn't change when you drag E, then adjust its value with the slider bar till the condition holds.

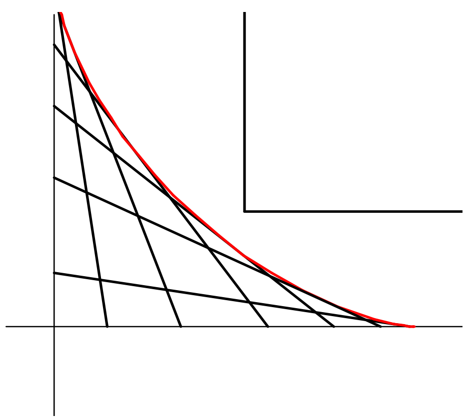

Figure 5: ladder positions are tangent to the envelope curve.

Display the implicit equation of the envelope (Figure 4).

Can you use the solve() command of a computer algebra system to find a value for L such that the point A lies on the envelope curve?

You'll find you get more than one solution. Can you identify the solution which corresponds to a real physical ladder length (can you have a negative length ladder? An imaginary length?)

Here's what Maple gives:

$$\frac{\sqrt{(y^2\,x)^{(1/3)}\left(y^2\,(y^2\,x)^{(1/3)}+3\,x\,(y^2\,x)^{(2/3)}+3\,y^2\,x+x^2\,(y^2\,x)^{(1/3)}\right)}}{(y^2\,x)^{(1/3)}}$$

The above expression is not as simple as it can be. Here is what Maple gives, so long as it is given the information that x>0 and y>0:

> simplify(%) assuming x>0,y>0;

$$(y^{(2/3)}+x^{(2/3)})^{(3/2)}$$

Can you verify that this is correct by hand?

Here is a helpful hint:

> expand((x^(2/3)+y^(2/3))^3);

$$x^2+3\,x^{(4/3)}\,y^{(2/3)}+3\,x^{(2/3)}\,y^{(4/3)}+y^2$$

Try replacing L with this value and they verify that with this length the corner lies on the envelope curve and this is the critical length of the ladder (Figure 6)

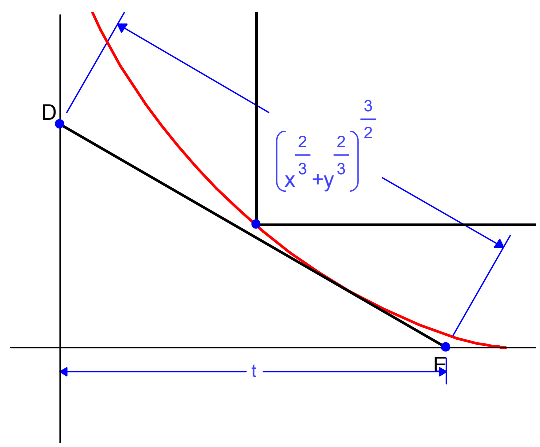

Figure 6: Solution to the longest ladder problem

If x=y, what does this expression simplify down to. Does this correspond to the solution we already have for the equal width case?